

## MAXWELL-LIKE RELATIONS IN CONDENSED MATERIALS. DECAY OF SHOCK WAVES†

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**Abstract**—The processes of shock wave decay are shown to include geometric attenuation, which is present for all wave amplitudes, hydrodynamic attenuation which is important only for large amplitude waves and depends only on mechanical and thermodynamic variables, and 'Maxwell attenuation' which results from the introduction of additional variables into the constitutive equations. A general shock wave decay equation is derived and illustrated by several examples. The effect of entropic flow behind the shock front is shown to modify the decay rate, but not to introduce new terms. The case of a viscous fluid is examined and shown to produce attenuation in addition to hydrodynamic attenuation. The relative importance of the new term has not been estimated.

### 1. INTRODUCTION

Traditional discussions of constitutive relations are fragmented and incoherent. Some progress has been made in recent years towards the development of general theories which encompass certain classes of constitutive relations [1, 2], but by and large most of us still go to thermodynamics for equations of state, fluid mechanics for the properties of viscous fluids, plasticity sources and dislocation theory for deformation, etc. One of the reasons for such fragmentation is the natural inclination of scientists to focus their attention on the subject of immediate interest, and the division of science is such that investigators interested in static phenomena, for example, are rarely concerned with real-time behavior of materials, plasticians rarely care about thermodynamic properties, etc. A major reason that this isolation has diminished in the last twenty years has been the rising interest in non-linear stress waves—principally shock waves—which dates from the period immediately following World War II.

Shock wave compressions can be very large and they are accompanied by high temperatures, so thermodynamic considerations are essential for understanding shock experiments and for using them to investigate material behavior. As soon as shock wave experimentation became sophisticated enough that wave profiles and their evolution in time could be recorded, it was evident that shock propagation phenomena resulted from a combination of thermodynamic and various irreversible, including time-dependent, properties. As a consequence, much of the effort in shock wave research during the last 15 years has been directed toward the understanding of these combined thermodynamic and non-thermodynamic effects and their use in the unravelling of various properties of materials in which they are made to propagate. A further development, as noted above, has been a deepening interest of theoreticians in the general theory of constitutive relations and their consequent involvement in the interpretation of experiments.

The central problem in which both thermodynamic and irreversible or time-dependent phenomena play a role is that of shock wave decay. The term 'decay' is used in a generic sense. Reactive or hydrodynamic processes may be such as to cause the shock amplitude to grow with time, but the equations are unchanged. A uniform shock state is useful for measurement of fundamental thermodynamic properties, but it yields no information about time-dependent properties. The steady shock transition region or 'permanent regime' contains such information, but it has proved to be of little use in analyzing the properties of solids. If, however, the shock wave profile can be recorded in a region where it is changing as it propagates, it may be possible to relate such changes to both reversible and irreversible material properties. In the most general case, both the time duration of application of the shock-producing pressure and its release may be controlled [3]; then both shock formation and shock decay can be studied. In the simple case, to which the present report is limited, pressure

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application is practically instantaneous and is followed by time-dependent release of pressure. The resulting shock wave decays as it propagates and the rate of change of shock amplitude with distance depends upon two additive terms:

$$\frac{Dp_1}{DX} = -A_1 \frac{\partial p}{\partial x} - A_2. \quad (1.1)$$

Here the shock has been assumed to be plane, with amplitude  $p_1(X)$  located at  $X(t)$  and to propagate in the direction of increasing  $x$ ;  $\partial p/\partial x$  is the gradient of shock pressure immediately behind the shock front,  $A_1$  depends on thermomechanical parameters and  $A_2$  depends on other, sometimes irreversible, properties of the material through which the shock is propagating, or of the system in which it is incorporated. For diverging waves a third term due to geometric attenuation appears on the right-hand side of equation (1.1).

The effect of the  $\partial p/\partial x$  term has often been called "hydrodynamic attenuation", since it is entirely independent of dissipative properties in the material. Existence of the second term,  $A_2$ , has denoted what has been called a "stress-relaxing solid", "relaxing solid", "viscoelastic solid", or "Maxwell solid". In this paper the material will be called "Maxwell-like" if  $A_2 \neq 0$ , whatever its source. In contrast with the effect of the first term, the effect of the  $A_2$  term will be called "Maxwell attenuation" to indicate that it depends on other than normal thermodynamic properties, i.e. stress, strain and entropy or temperature. In the following sections the natures of hydrodynamic and Maxwell attenuations will be illustrated, the coefficients  $A_1$  and  $A_2$  will be derived under rather general conditions and several examples will be given in which  $A_2$  is non-vanishing.

The main objective of this paper is to demonstrate that whenever the (adiabatic) constitutive relation for the shocked state involves more than two parameters, the material will be Maxwell-like.

## 2. MECHANISMS OF SHOCK DECAY

### *Stress and strain conventions*

We shall be dealing exclusively with waves in one space dimension,  $x$ . All equations will refer to mechanically-isotropic materials. Expansion to anisotropic media can be accomplished by procedures described elsewhere [4, 5]. Materials of interest are primarily solids or viscous fluids, so shear stresses commonly exist. Diagonal stress components will be exclusively compressive, so it is convenient to follow the practice of fluid dynamics and use the pressure tensor,  $p_{ij}$ , which is the negative of the stress tensor,  $\sigma_{ij}$ , commonly used in solid body mechanics:

$$p_{ij} = -\sigma_{ij}$$

With the coordinate convention described above,  $x$ ,  $y$  and  $z$  are principal coordinates; off-diagonal components of  $p_{ij}$  vanish, and diagonal components can be described by a single subscript;  $p_x \equiv p_{xx}$ ,  $p_y \equiv p_{yy}$ ,  $p_z \equiv p_{zz}$ . Because of the symmetry of one-dimensional waves,  $p_y = p_z$ . No motions parallel to wave fronts will be considered, so the only non-vanishing component of strain for plane waves is  $\epsilon_x \equiv \epsilon_{xx}$ ; this is called a "state of uniaxial strain".

The only pressure component which is normally measured in experiments with plane shock waves is  $p_x$ , which can be looked upon as composed of mean pressure,  $\bar{p}$ , and a shearing stress,  $\tau$ . This useful resolution is a simple identity:

$$\begin{aligned} p_x &= (p_x + 2p_y)/3 + 2(p_x - p_y)/3 \\ &= \bar{p} + 4\tau/3 \end{aligned}$$

where  $\tau$  is shear stress on planes with normals at  $45^\circ$  to the  $x$  axis. When dealing with hydrodynamic states, or when shock wave results are to be compared with static measurements,  $\bar{p}$  will be identified with hydrostatic pressure  $P$ .



### Hydrodynamic attenuation

The nature of hydrodynamic attenuation was clearly illustrated in a simple way by Sir Geoffrey Taylor in 1939[6]. In Fig. 1 are shown pressure-distance profiles of a shock wave in a fluid at two successive times. The velocity of propagation of a point  $Q$  immediately behind the shock front is the local sound velocity,  $c$ , plus the particle velocity,  $u$ , with which it is carried along behind the shock. It is readily shown that  $u + c > R$ , the propagation velocity of the shock itself. As the rarefaction overtakes the shock front, the shock amplitude is diminished. If we assume that the diminution in shock amplitude which occurs when point  $Q$  overtakes the shock front is exactly equal to  $-\Delta s \partial P/\partial x$ , where the derivative is evaluated immediately behind the shock front, we can readily determine the rate of decay of the shock. Point  $Q$  travels the distance  $\Delta s + \Delta X$  in the same time it takes the shock front to travel the distance  $\Delta X$ , i.e.

$$\Delta t = \frac{\Delta X}{R} = \frac{\Delta X + \Delta s}{u + c}.$$

For  $\Delta p = -\Delta s \partial p/\partial x$ , this gives

$$\frac{\Delta P}{\Delta X} \rightarrow \frac{DP}{DX} = -\frac{(u + c - R)}{R} \frac{\partial P}{\partial x}. \quad (2.1)$$

The difference  $u + c - R$  increases monotonically with the curvature of the Rankine-Hugoniot  $p$ - $v$  curve, so equation (2.1) shows that shock decay is rapid where the thermodynamic derivative  $\partial^2 p/\partial v^2$  is large in the shocked state and  $\partial p/\partial x$  is large, i.e. the shock is a sharp spike.

A. J. Harris in 1942 and 1943 derived an exact equation for decay of shock waves in fluids for plane, cylindrical and spherical geometries, assuming that the shock front is discontinuous and flow behind the shock is isentropic[7, 8]. His result for a fluid with an arbitrary equation of state is [8]:

$$R \left( R - u + \frac{c^2}{v} \frac{du}{dp} \right) \frac{DP}{DX} = [(R - u)^2 - c^2] \frac{\partial P}{\partial x} - \frac{(n - 1)c^2 u (R - u)}{vX} \quad (2.2)$$

where  $n = 1$  for plane waves, 2 for cylindrical waves and 3 for spherical waves,  $v$  is specific volume, and  $X$  is the position of the shock front. When  $n = 2$  or 3, the second term on the right provides 'geometric attenuation', which exists even for infinitesimal waves. The first term on the right-hand side of equation (2.2) vanishes like  $u^2$ , so it is negligible for infinitesimal waves.

Equation (2.2) with  $n = 1$  contains terms in addition to those in equation (2.1). In the approximation of equation (2.1), the reflected wave produced when the overtaking rarefaction reaches the shock front is neglected. The amplitude of this reflected wave is of the order  $u^3$ , so equations (2.1) and (2.2) differ sensibly only for strong shocks.

Maxwell attenuation can be illustrated by a simple relaxing fluid. The relation between pressure and density under isentropic conditions for a non-relaxing fluid is  $dP/d\rho = c^2$  or  $\dot{P} = c^2 \dot{\rho}$ , where the dot denotes the convective derivative and  $c$  is sound velocity. In this case the equations of continuity and motion can be combined to give a pair of equations for

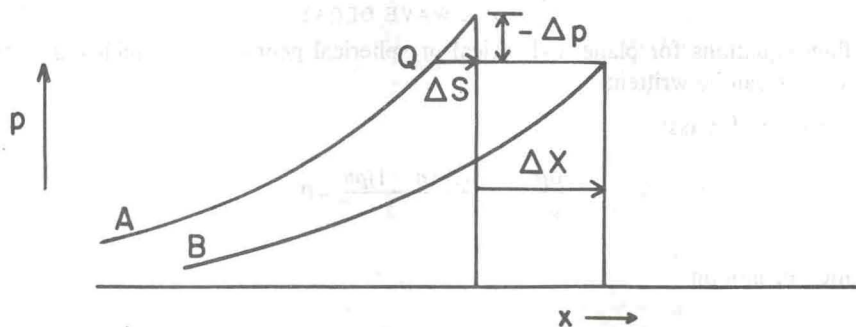


Fig. 1. Decay of a plane shock wave. A and B are wave profiles at times  $t$  and  $t + \Delta t$ , respectively.  $\Delta t$  = time required for the point  $Q$  to overtake the shock front.



travelling waves [9]. For waves propagating in the +x direction, this equation is

$$\frac{D(u+l)}{Dt} = 0 \quad (2.3)$$

where  $D/Dt = \partial t + (u+c) \partial/\partial x$ , and  $l = \int dP/\rho c$ . A statement equivalent to equation (2.3) is that  $u+l = \text{const.}$  on every  $C_+$  characteristic.

Suppose now that every incremental change in pressure in an element of the fluid is followed by relaxation to some equilibrium state, say  $P_s(\rho)$ . Then the relation between  $P$  and  $\rho$  has the form

$$\dot{P} = c^2 \dot{\rho} - F(P, \rho) \quad (2.4)$$

where  $F(P, \rho)$  is the relaxation function. For example, a simple approximation to  $F$  might be

$$F = |P - P_s(\rho)|/T \quad (2.5)$$

where  $T$  is a constant relaxation time.

When equation (2.4) is combined with the plane flow equations, equation (2.3) is replaced by

$$\frac{D}{Dt}(u+l) = -F/\rho c. \quad (2.6)$$

For small disturbances  $l = \Delta P/\rho c$  with  $\rho$  and  $c$  approximately constant. Moreover,  $u \approx \Delta P/\rho c$ , so equation (2.6) becomes

$$D(\Delta P)/Dt \approx -F/2 \quad (2.7)$$

or

$$D(\Delta P)/Dx \approx -F/2c \quad (2.8)$$

where  $c$  is the propagation velocity of small disturbances. Equation (2.7) is analogous to equations which describe other decay processes, say the decay of a radioactive population,  $N$ :  $dN/dt = -\lambda N$ . The difference is that the time derivative in this case is a directional derivative along the path of wave propagation. If one considers a layer of finite thickness,  $\Delta x$ , the wavefront takes a finite time,  $\Delta x/c$ , to cross the layer, and during this time the wave amplitude decays an amount  $F\Delta x/2c$ . However small  $\Delta x$  may be, this decay occurs and accumulates from one layer to the next; or, from equation (2.4), if a mass element is out of equilibrium, i.e.  $P \neq P_s$ ,  $P$  undergoes a reversible change  $c^2 \dot{\rho} \Delta t$  in time  $\Delta t$ , and in addition it changes by  $-F\Delta t$ . The latter change occurs even if  $\dot{\rho} = 0$ .

It will be shown in the next section how geometric, hydrodynamic and Maxwell attenuation combine to produce the net decay of a shock wave in a solid whenever pressure in the shock depends on variables other than material density.

### 3. SHOCK WAVE DECAY

The flow equations for plane, cylindrical or spherical geometry in which only one space variable occurs can be written:

Conservation of mass:

$$\frac{d\rho}{dt} + \rho \frac{\partial u}{\partial x} + \frac{(n-1)\rho u}{x} = 0. \quad (3.1)$$

Equation of motion:

$$\rho \frac{du}{dt} = -\frac{\partial p_x}{\partial x} - \frac{2(n-1)\tau}{x}. \quad (3.2)$$

Conservation of energy:

$$\frac{de}{dt} = \frac{dw}{dt} + \frac{dq}{dt} \quad (3.3)$$

Here  $\rho \equiv 1/v$  is material density,  $u$  is material velocity,  $e$  is specific internal energy,  $dw/dt$  is the rate at which work is done on unit mass,  $dq/dt$  is the rate at which heat is delivered to unit mass,  $d/dt$  denotes the convective derivative,  $n = 1, 2$  and  $3$  for plane, cylindrical and spherical waves, respectively.

If the convective derivative of entropy is small immediately behind the shock front, equations (3.2) and (3.3) are redundant. Consider this case first and suppose that

$$p_x = p_x(v, \xi) \quad (3.4)$$

where  $\xi$  is an additional physical variable on which  $p_x$  depends. It might, for example, be plastic strain, strain rate or electric field. Then

$$\frac{dp_x}{dt} = \frac{\partial p_x}{\partial v} \frac{dv}{dt} + \frac{\partial p_x}{\partial \xi} \frac{d\xi}{dt} = a^2 \frac{d\rho}{dt} + \alpha \frac{d\xi}{dt} \quad (3.5)$$

where  $a$  is frozen sound speed, i.e. sound speed with  $\xi = \text{constant}$ . Elimination of  $d\rho/dt$  between equations (3.1) and (3.5) gives

$$\frac{dp_x}{dt} + a^2 \rho \frac{\partial u}{\partial x} - \alpha \frac{d\xi}{dt} + \frac{\rho u a^2 (n-1)}{x} = 0. \quad (3.6)$$

Denote path of the shock front by  $x = X(t)$  and shock velocity by  $R = DX/Dt$ . Derivative of any field variable  $f(x, t)$  along a path parallel to the shock front is denoted  $Df/Dt$ :

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + R \frac{\partial f}{\partial x} = \frac{df}{dt} + (R - u) \frac{\partial f}{\partial x} \quad (3.7)$$

since  $df/dt = \partial f/\partial t + u \partial f/\partial x$ . Substitution of equation (3.7) into equations (3.2) and (3.6) gives the following pair:

$$\frac{Du}{Dt} - (R - u) \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p_x}{\partial x} - \frac{2(n-1)u}{\rho x} \quad (3.8)$$

$$\frac{Dp_x}{Dt} + a^2 \rho \frac{\partial u}{\partial x} = (R - u) \frac{\partial p_x}{\partial x} + \alpha \frac{d\xi}{dt} - \frac{\rho u a^2 (n-1)}{x}. \quad (3.9)$$

Now apply equations (3.8) and (3.9) to the region just behind the discontinuity representing the shock. The shock jump condition which represents the equation of motion is

$$p_x = \rho_0 R u, \quad (3.10)$$

where pressure in the unshocked state is assumed to be negligible and  $\rho_0$  denotes unshocked mass density. Any change in shock pressure  $p_x$  is accompanied by changes in  $R$  and  $u$ :

$$\frac{1}{p_x} \frac{Dp_x}{Dt} = \frac{1}{R} \frac{DR}{Dt} + \frac{1}{u} \frac{Du}{Dt} \quad (3.11)$$

$$= \frac{A}{R} \frac{Dp_x}{Dt} + \frac{B}{R} \frac{D\xi}{Dt} + \frac{1}{u} \frac{Du}{Dt} \quad (3.12)$$

where

$$A = \frac{\partial R}{\partial p_x}, \quad B = \frac{\partial R}{\partial \xi} \quad (3.13)$$



Equation (3.12) can be used to eliminate  $Du/Dt$  from equation (3.8). The result is

$$\left(\frac{u}{\rho_x} - \frac{uA}{R}\right) \frac{Dp_x}{Dt} - (R-u) \frac{\partial u}{\partial x} = \frac{uB}{R} \frac{D\xi}{Dt} - \frac{1}{\rho} \frac{\partial p_x}{\partial x} - \frac{2(n-1)\tau}{\rho X}. \quad (3.14)$$

It is now possible to eliminate  $\partial u/\partial x$  between equations (3.9) and (3.14):

$$\left[(R-u) + \frac{a^2 \rho u}{\rho_x} - \frac{a^2 \rho Au}{R}\right] \frac{Dp_x}{Dt} = [(R-u)^2 - a^2] \frac{\partial p_x}{\partial x} - \frac{2a^2(n-1)\tau}{X} - \frac{\rho a^2 u(R-u)(n-1)}{X} + \alpha(R-u) \frac{d\xi}{dt} + \frac{a^2 \rho Bu}{R} \frac{D\xi}{Dt}. \quad (3.15)$$

With  $D\xi/Dt = d\xi/dt + (R-u) \partial \xi/\partial x$ , equation (3.15) becomes

$$\frac{Dp_x}{Dt} = M \frac{\partial p_x}{\partial x} + L \frac{d\xi}{dt} + N \frac{\partial \xi}{\partial x} - \frac{G}{X} \quad (3.16)$$

where

$$M = (R-u)[(R-u)^2 - a^2]/Q,$$

$$L = [\alpha(R-u)^2 + a^2 \rho_0 Bu]/Q,$$

$$N = a^2 \rho_0 Bu(R-u)/Q,$$

$$Q = (R-u)^2 + a^2(1 - \rho_0 Au),$$

$$G = a^2(n-1)(\rho_x + 2\tau)(R-u)/Q.$$

If  $R$  and  $u$  depend only on  $p_x$ ,  $B \equiv \partial R/\partial \xi = 0$ , and  $A = dR/dp_x = (1 - \rho_0 R du/dp_x)/\rho_0 u$ . Using the identity  $\rho(R-u) = \rho_0 R$ , we find

$$1 - \rho_0 Au = \rho_0 R du/dp_x. \quad (3.17)$$

Divide equation (3.16) by  $R$  to obtain  $Dp_x/DX$ . Then with  $\xi = \text{const.}$ ,  $a^2 = c^2$ . Set  $\tau = 0$ ; use equation (3.10), the shock jump condition  $\rho(R-u) = \rho_0 R$  and equation (3.17) in equation (3.16) and it reduces to the Harris relation, equation (2.2). The effect of finite strength, represented by  $\tau$ , is to increase the rate of geometric attenuation.

It may happen that  $R$  is very insensitive to  $\xi$ , so that the coefficient  $N$  vanishes, but  $L$  is still sensible. Then Maxwell attenuation proportional to  $d\xi/dt$  will exist.

### Examples

(i) *Elastic-plastic solids.* In an elastic-plastic-relaxing solid, outside the yield surface,  $p_x$  depends on both  $v$  and plastic strain,  $\epsilon_x^p$ . If stresses are supported by elastic strains alone, and if plastic dilatation vanishes [10],

$$\begin{aligned} \dot{p}_x &= a^2 \dot{\rho} - 2\mu \dot{\epsilon}_x^p \\ &\equiv a^2 \dot{\rho} - F \end{aligned} \quad (3.18)$$

where  $a^2$  is independent of  $\epsilon_x^p$  and  $F$  is the relaxation function. With  $\xi = \epsilon_x^p$ ,  $\alpha = -2\mu$ , and  $B = 0$ , equation (3.16) becomes

$$\frac{Dp_x}{Dt} = M \frac{\partial p_x}{\partial x} - \frac{2\mu(R-u)^2 \dot{\epsilon}_x^p}{(R-u)^2 + a^2(1 - \rho_0 Au)}. \quad (3.19)$$

(ii) *Piezoelectric solids.* In an axial mode piezoelectric device a plane shock is made to propagate in the direction of polarization and a depolarization current,  $I$ , is produced in an external circuit. If  $p_x$  is allowed to depend on both  $\rho$  and electric displacement,  $D$ ,  $\xi$  in equation



(3.5) is replaced by  $D$  and  $\alpha = -e_{11}/\epsilon_{11}^s$  [11]. For most materials  $dR/dD$  will be negligible. Moreover, for weak shocks  $\rho_0 R du/dp \approx 1$ ; then if  $M \partial p_x / \partial x$  is negligible, as is often the case, equation (3.16) becomes

$$\frac{Dp_x}{Dt} = -\frac{e_{11}/\epsilon_{11}^s}{1 + a^2/(R-u)^2} \frac{dD}{dt} \approx -\frac{e_{11}}{2\epsilon_{11}^s} \frac{dD}{dt}.$$

For the kind of device assumed, current in the external circuit,  $I(t)$ , is proportional to  $dD/dt$ , so

$$\frac{Dp_x}{Dt} \approx -e_{11}I(t)/2C\epsilon_{11}^s \quad (3.20)$$

where  $C$  is cross-section area of the shocked material [12]. If the external load is resistive,  $I(t) > 0$  and the analogy with the relaxing solid is close. If the external load has a resonance,  $I(t)$  may alternate signs, causing an analogous oscillation in the decay rate.

(iii) *Viscous fluids.* For a Newtonian fluid with shear viscosity  $\nu$ ,

$$p_x = p(v) + 4\nu\dot{\epsilon}_x/3 \quad (3.21)$$

where  $\xi = \dot{\epsilon}_x$  and  $\alpha = 4\nu/3$ . Equation (3.5) becomes

$$\dot{p} = \rho c^2 \dot{\xi}_x + 4\nu\ddot{\xi}_x/3 \quad (3.22)$$

where  $c^2$  is the hydrodynamic sound speed and  $\dot{\xi}_x = (1/\rho) d\rho/dt$ .

This is an interesting case because the form of equation (3.21) precludes the possibility of a discontinuous shock front. Were this to occur,  $\dot{\epsilon}_x$ , and therefore  $p_x$ , would become infinite, which is a contradiction. The jump conditions apply also to a steady transition connecting two uniform states, but that case is not interesting in the present context. If the shock transition is followed by a rarefaction, the rarefaction and shock interfere in some degree, so the jump conditions are no longer exact. This is apparent from equation (3.22), which shows that  $p$  and  $\rho$  do not simultaneously achieve their maximum values. The locus of states connecting shock and rarefaction does not, therefore, form a cusp. Instead, it is a continuous convex curve connecting the shock line and the expansion isentrope. Moreover, the entire problem of defining a propagation velocity is reopened and cannot readily be settled in a satisfactory way [24]. However, numerical calculations have shown that under such conditions as will exist, the lower part of the shock profile will still closely conform to the steady profile. Also the shock jump conditions still describe the relations among propagation velocity, peak pressure, particle velocity and density reasonably well. In other words, the locus of  $p$ ,  $v$  states through even a non-steady shock will not deviate very much from the Rayleigh line, except for very large viscosities. The pressure profile in this unsteady quasi-shock will be somewhat as shown in Fig. 2. If the shock jump conditions are used to connect state  $A$  and the initial state, equation (3.16) applies. With  $B = 0$  and the approximations  $R - u \approx c$  and  $dp_x/du = R$ , it becomes

$$\frac{Dp_x}{Dt} \approx (R - u - c) \frac{\partial p_x}{\partial x} + \frac{2\nu}{3} \ddot{\xi}_x \quad (3.23)$$

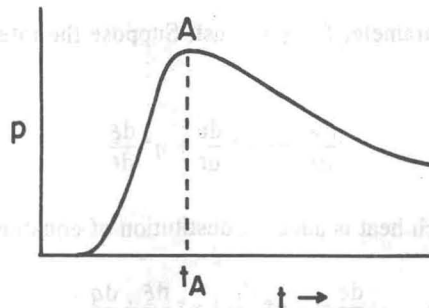


Fig. 2. Schematic representation of pressure-time profile of a shock in a viscous medium.



Using equation (3.7), equation (3.23) becomes

$$\frac{Dp_x}{Dt} \approx \left(1 + \frac{u-R}{c}\right) \dot{p}_x + \frac{2\nu}{3} \ddot{\epsilon}_x \quad (3.24)$$

At  $A$ ,  $\dot{p}_x = 0$ , and from equation (3.22)  $2\nu\ddot{\epsilon}_x/3 = -\rho c^2 \dot{\epsilon}_x^A/2$ . That is, the rate of decay of the shock depends on strain rate at  $A$ , and this in turn depends directly on the viscosity,  $\nu$ . This result is in marked contrast with that of equations (2.1) and (2.2) and is not easily explicable.

If the peak of the  $\rho(t)$  profile is chosen, instead of  $p_x(t)$ , the situation is altered. Now  $\dot{\epsilon} = 0$  and  $\dot{p}_x = 4\nu\ddot{\epsilon}_x^A/3$  and equation (3.24) becomes

$$\begin{aligned} \frac{Dp_x}{Dt} &\approx \left(\frac{3}{2} + \frac{u-R}{c}\right) \left(\frac{dp_x}{dt}\right)_{\dot{\epsilon}_x=0} \\ &= \left(\frac{3}{2} + \frac{u-R}{c}\right) \cdot \frac{4\nu}{3} (\ddot{\epsilon}_x)_{\dot{\epsilon}_x=0} \\ &\approx 2\nu(\ddot{\epsilon}_x)_{\dot{\epsilon}_x=0} \end{aligned} \quad (3.25)$$

Since characteristics in the rarefaction following the shock are still expected to overtake the shock and to provide something approximating the usual hydrodynamic attenuation, it seems likely that the viscosity term produces an additional attenuation. Whether or not this additional attenuation is significant remains to be determined.

(iv) *Plastic shear-yielding solids*. If, in this case, a shear stress,  $\tau$ , proportional to plastic strain rate exists and there is no plastic dilatation,

$$\tau = 2\nu(\dot{\epsilon}_x^p - \dot{\epsilon}_y^p)/2 \quad p_x = p_s(v) + 2\nu\dot{\epsilon}_x^p$$

where  $p_s(v)$  represents the quasi-static dependence of  $p$  on  $v$ . Now  $\xi = \dot{\epsilon}_x^p$ ,  $\alpha = 2\nu$ , and  $B = 0$ . The problem is exactly analogous to that for a viscous fluid.

Combinations of the above effects are possible. Extension of the analysis to include two or more variables,  $\xi_1, \xi_2$ , etc. is straightforward. Similar relations have been used to describe wave propagation in chemically-reacting media[13, 14]. Coleman, Chen and others have obtained similar equations for a variety of problems[15-18].

#### 4. THERMAL EFFECTS

Suppose that flow behind the shock is entropic and that  $\xi = S$ , the specific entropy. Equation (3.16) follows, as before, but it now contains derivatives of  $S$ . These can be eliminated through use of equation (3.3) and some assumptions about the underlying thermodynamics of the material.

Let

$$p_x^* = p_x^*(v, S, \xi).$$

Then

$$\frac{dp_x^*}{dt} = a^{*2} \frac{d\rho}{dt} + \frac{\Gamma^* T}{v} \frac{dS}{dt} + a^* \frac{d\xi}{dt} \quad (4.1)$$

where  $\Gamma$  is the Grüneisen parameter for  $\xi = \text{const}$ . Suppose the rate at which work is done on unit mass is

$$\frac{dw}{dt} = -p_x^* \frac{dv}{dt} + \eta^* \frac{d\xi}{dt} \quad (4.2)$$

and  $dq/dt$  is the rate at which heat is added. Substitution of equation (4.2) into (3.3) gives

$$\frac{de}{dt} = -p_x^* \frac{dv}{dt} + \eta^* \frac{d\xi}{dt} + \frac{dq}{dt} \quad (4.3)$$



If internal energy is a state function of  $v$ ,  $S$ ,  $\xi$ , then

$$\frac{de}{dt} = \left. \frac{\partial e}{\partial v} \right|_{S, \xi} \frac{dv}{dt} + \left. \frac{\partial e}{\partial S} \right|_{v, \xi} \frac{dS}{dt} + \left. \frac{\partial e}{\partial \xi} \right|_{v, S} \frac{d\xi}{dt} \quad (4.4)$$

$$\equiv -p_x \frac{dv}{dt} + T \frac{dS}{dt} + \eta \frac{d\xi}{dt}. \quad (4.5)$$

If  $\xi$  is a dissipative variable,  $e$  may still be a state function of  $v$  and  $S$  and  $\eta = 0$ . Eliminating  $de/dt$  between equations (4.3) and (4.5) yields

$$T \frac{dS}{dt} = (p_x - p_x^*) \frac{dv}{dt} + (\eta^* - \eta) \frac{d\xi}{dt} + \frac{dq}{dt}. \quad (4.6)$$

Entropy,  $S$ , can now be eliminated between equations (4.1) and (4.6):

$$[a^{*2} + (p_x^* - p_x)v\Gamma^*] \frac{d\rho}{dt} = \frac{dp_x^*}{dt} - [\alpha^* + \rho\Gamma^*(\eta^* - \eta)] \frac{d\xi}{dt} - \rho\Gamma^* \frac{dq}{dt}. \quad (4.7)$$

If the system is reversible with  $p^* = p$ ,  $\eta^* = \eta$ , and if  $dq/dt = 0$ ,  $\alpha^* = \alpha$ ,  $a^* = a$ , equation (4.7) reduces to equation (3.5). The calculation proceeds as before from this point; the difference is that the extra terms of equation (4.7) must be carried in the computation. The extension to include several variables  $\xi_1, \xi_2$ , etc. is straightforward.

#### Examples

(i) *Elastic-plastic relaxing solids*. The work of uniaxial compression can be expressed as

$$dw = v p_x d\epsilon_x. \quad (4.8)$$

If elastic and plastic strains are occurring simultaneously, increments in elastic and plastic strain may be assumed to be additive:

$$d\epsilon_x = d\epsilon_x^e + d\epsilon_x^p \quad (4.9)$$

where superscripts  $e$  and  $p$  stand for elastic and plastic, respectively. It can also be assumed, to a good approximation for many substances, that there is no density change associated with plastic strain:

$$d\Theta = d\Theta^e + d\Theta^p = d\Theta^e = \sum_i d\epsilon_{ii}^e = d\epsilon_x = d\rho/\rho. \quad (4.10)$$

The pressure and strain deviators are, respectively,

$$d\Pi_{ij} = dp_{ij} - \delta_{ij} d\bar{p}, \quad d\epsilon_{ij} = d\epsilon_{ij} - \delta_{ij} d\Theta/3, \quad d\epsilon_{ij} = d\epsilon_{ij}^e + d\epsilon_{ij}^p. \quad (4.11)$$

Take principal axis coordinates with uniaxial compression along the  $x$ -axis. Then increments in the work of elastic and plastic deformation are

$$dw_{de} = v \sum_j \Pi_j d\epsilon_j^e \quad \text{elastic}, \quad (4.12)$$

$$dw_{dp} = v \sum_j \Pi_j d\epsilon_j^p \quad \text{plastic}. \quad (4.13)$$

It is plausible to assume that stress is supported entirely by the elastic strains. To see this, consider the microscopic behavior of a plastically-deforming material from the point of view of dislocation theory, where plastic deformation is synonymous with motion and generation of



dislocations. In most materials dislocations probably move at all stress levels, so there is no such thing as a yield point and there is no truly elastic behavior. But in practice the yield point concept is useful and the yield point itself may be taken to be the point at which large numbers of dislocations are set in motion. The motion of dislocations is inhibited by the existence of energy barriers which must be overcome by combination of stored elastic energy and thermal fluctuations of atoms. So, for example, one may consider a strained, work-hardening solid as one in which many dislocations exist but are momentarily immobilized or 'pinned' by energy barriers. If applied stress is held constant, dislocations may occasionally overcome a barrier and move to the next barrier, thus contributing to the plastic deformation by creep.

If the external stress is increased, local strain energy is increased around the pinning points, more dislocations are moved through pinning points and pass on until they are pinned again. If it is assumed that dislocations move freely between pinning points, the plastic strain which results from their motion requires no part of the applied stresses; i.e. the applied stress is supported entirely by the elastic part of the strain. It does follow, however, that a relaxation process may exist. For example, an increment in stress increases local strain energy around pinning points and increases probability that dislocations will break free of their pinning points and move on until pinned again. However, the probability of a dislocation breaking free is time dependent because it depends on thermal fluctuations as well as strain energy. This means that an increment in stress will produce an immediate elastic strain, and that, as time passes, dislocations escape their pins, move and are repinned, so that a plastic strain develops. Or the original increment in strain is reduced as the pinned dislocations escape from their barriers and the elastic strain is reduced. This concept can be expressed quantitatively in the following way for uniaxial strain in an isotropic solid:

$$\begin{aligned} dp_x &= \lambda d\Theta + 2\mu d\epsilon_x^e \\ &= (\lambda + 2\mu) d\epsilon_x - 2\mu d\epsilon_x^p, \end{aligned} \quad (4.14)$$

when divided by  $dt$  this yields equation (3.18). In terms of stress and strain deviators this translates to

$$\begin{aligned} \dot{\Pi}_j &= 2\mu \dot{e}_j - 2\mu \dot{e}_j^p \\ &= 2\mu \dot{e}_j - F(S, e) \end{aligned} \quad (4.15)$$

where  $F$  is a relaxation function.

Equations (4.14) and (4.15) are both based on the assumption that dislocations move between pinning points without drag. If this is not true, a viscous contribution to the stress appears:

$$\begin{aligned} dp_x &= (\lambda + 2\mu) d\epsilon_x - 2\mu d\epsilon_x^p + \eta d\dot{\epsilon}_x^p \\ \dot{\Pi}_j &= 2\mu \dot{e}_j - 2\mu \dot{e}_j^p + 2\eta \dot{e}_j^p. \end{aligned}$$

For uniaxial strain

$$\begin{aligned} \Pi_x &= 4\tau/3, & \Pi_y &= \Pi_z = -2\tau/3, \\ e_x &= 2\epsilon_x/3, & e_y &= e_z = -\epsilon_x/3. \end{aligned}$$

Then

$$\begin{aligned} d\Pi_x &= 4d\tau/3 = 2\mu d\epsilon_x^e, \\ d\Pi_y &= -2d\tau/3 = 2\mu d\epsilon_y^e, \\ d\tau &= 2\mu d\gamma^e, \end{aligned}$$

where

$$d\gamma^e \equiv d(e_x^e - e_y^e)/2 = (d\epsilon_x^e - d\epsilon_y^e)/2.$$



The incremental work of total deformation is

$$dw_d = 4v\tau d\epsilon_x/3. \quad (4.16)$$

The incremental work of elastic deformation is

$$dw_{de} = 8v\tau d\gamma^e/3 = (4v\tau/3\mu) dt = (2v/3\mu) d(\tau^2). \quad (4.17)$$

The incremental work of plastic deformation is obtained by subtracting equation (4.17) from equation (4.16):

$$dw_p = (4/3)v\tau(d\epsilon_x - d\tau/\mu) = (8/3)v\tau d\gamma^p \quad (4.18)$$

where  $d\gamma^p = (d\epsilon_x^p - d\epsilon_y^p)/2$ .

If we assume that there is always an elastically-strained state imbedded in every deformed state, and that this elastic state is inherently reversible and recoverable, it follows that an internal energy function exists which depends on elastic strains, entropy, and perhaps internal variables. Neglecting internal variables, increments in the internal energy function can be written [19]

$$de = T dS + dw_e. \quad (4.19)$$

According to equation (4.19) and the foregoing assumptions, internal energy can be expressed in terms of entropy and the elastic strains. For example,

$$e = e(S, \epsilon_x, \epsilon_x^p). \quad (4.20)$$

The appearance of  $\epsilon_x^p$  in this potential is illusory. It does not imply a physical dependence of  $e$  on plastic strain. Nonetheless,  $(\partial e/\partial \epsilon_x^p)_{\epsilon_x, S}$  exists and must be counted in applications.

Referring now to equation (4.1), we have  $\xi = \epsilon_x^p$ ,  $p_x = p^*$ ,  $a^* = a$ ,  $I^* = (\partial p_x/\partial S)_{\rho, \epsilon_p}$ ,  $\alpha^* = -2\mu$ , where  $\mu$  is the rigidity modulus. Comparing equations (4.2) and (4.8) we have  $\eta^* = 0$ . Equations (4.5) and (4.20) give  $\eta = -2v\tau$ . Then equation (4.7) becomes, with  $dq/dt = 0$ ,

$$a^2 \frac{d\rho}{dt} = \frac{dp_x}{dt} - 2\mu \left(1 - \frac{\Gamma^* \tau}{\mu}\right) \frac{d\epsilon_x^p}{dt}. \quad (4.21)$$

Since  $\tau$  is equal to half the yield stress, the effect of including the thermal variables is to slightly decrease the effective relaxation function [20].

(ii) *Phase transitions in liquids.* These have been treated in detail elsewhere [21, 22]. In the present formalism, if  $\lambda$  is the fraction of material in phase 2, the parameters of equation (4.1) are [23]:

$$a^{*2} = \left. \frac{\partial P}{\partial \rho} \right|_{S, \lambda} = v^2/(\beta v - T\gamma^2 v^2/C_{P\lambda}), \quad (4.22)$$

where

$$\begin{aligned} \gamma &= \frac{1}{v} \left( \frac{\partial v}{\partial T} \right)_{P, \lambda}, \\ C_{P\lambda} &= (1-\lambda)C_{P1} + \lambda C_{P2}, \\ C_{Pi} &= \text{sp. heat at constant pressure of phase } i, \\ \beta &= -\frac{1}{v} \left( \frac{\partial v}{\partial P} \right)_{T, \lambda}, \\ \Gamma^* &= \gamma v^2/(\beta v C_{P\lambda} - \gamma^2 v^2 T), \end{aligned} \quad (4.23)$$

$$\alpha^* = \left. \frac{\partial P}{\partial \lambda} \right|_{v, S} = \frac{\gamma v T \Delta S - C_{P\lambda} \Delta v}{\gamma^2 v^2 T - \beta v C_{P\lambda}} \quad (4.24)$$



Equation (4.2) is unchanged:  $\dot{w} = -P^*\dot{v}$ , i.e.  $\eta^* = 0$ . Equation (4.3) becomes

$$\dot{e} = -P\dot{v} + T\dot{S} + (\mu_2 - \mu_1)\dot{\lambda}$$

where  $\mu_1$  and  $\mu_2$  are the chemical potentials of phases 1 and 2. Then

$$\eta = \mu_2 - \mu_1. \quad (4.25)$$

In equation (4.7),  $p_x^* = p_x = P$ ,  $\eta^* = 0$ ,  $a^*$ ,  $\Gamma^*$ ,  $\alpha^*$  and  $\eta$  are given by equations (4.22)–(4.25).

## 5. DISCUSSION

Equations (3.5) and (4.7) show the effects of introducing a variable other than  $v$  and  $S$  in the constitutive equation of material through which a plane wave is passing. The consequence is always to add an additional term to the  $\dot{p}$ ,  $\dot{\rho}$ ,  $\dot{S}$  relation. This additional term is carried through the flow equations and invariably introduces 'Maxwell attenuation' into the shock decay process. The existence of this term does not depend on the additional process being dissipative, though its coefficient is enhanced by dissipation, which causes flow behind the shock to be entropic.

The fundamental reason for this effect lies in the fact that there are but three equations to describe continuum flow and four variables,  $p$ ,  $\rho$ ,  $S$ ,  $u$ . A single constitutive relation then provides a soluble set, as long as no new variable is introduced. If it is, it must be carried through the equations in an *ad hoc* way. Only if another constitutive relation and a corresponding differential equation are added, can this situation be avoided. For example, if electric displacement is added, as in equation (3.20), the Maxwell-like term can be eliminated if an additional constitutive relation and the appropriate electromagnetic equations are added.

When the extra variable is a density derivative, as for the viscous fluid, a curious situation arises. One can proceed, in principle, to combine the constitutive relation, equation (3.21), with the flow equations, eliminate  $p$  and  $u$ , and solve the resulting third order equation for  $\rho$ . But in so doing the utility of heuristic understanding of wave propagation problems is lost and the new problem stands in isolation. Yet one feels that the problem as stated is still essentially a wave propagation problem, and that only a little tinkering with the mathematical apparatus of wave propagation should provide understanding. This view may be wrong; certainly it has not yet yielded an estimate of the importance of viscous attenuation in the shock decay process or a satisfactory description of its interaction with hydrodynamic attenuation.

The reader should be aware that an extensive literature exists of which the principal purpose is to determine the entire propagation history of decaying shock waves, principally in gases. The problem has little relation to the one discussed here, but the interested reader can gain entry to the literature through Ref. [9, p. 160].

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## NOTE ADDED IN PROOF

It appears to be useful to look at the effect of viscosity on decay in the following way. In the case of a discontinuous shock followed by a rarefaction, the rate of hydrodynamic attenuation is determined by the difference between the square roots of the slopes of the Rayleigh line and the rarefaction isentrope where they meet on the Hugoniot curve. When viscosity is present, these two curves do not join in a cusp and the pertinent slopes are somewhat below the maximum value of  $p_x$ ; they are now to be taken where the true locus of states falls into the Rayleigh line and isentrope. The difference in slopes is then somewhat smaller than that for the sharp shock and the rate of decay will be somewhat reduced. It is not yet clear that this consideration is amenable to quantitative treatment.